Polynomial Division and Factor Theorem

Definition: A polynomial is a sum/difference of terms with **non-negative integer powers** of x. The **highest** power of x is its **degree**.

The following are polynomials
- x^3+4x^2-3x+1 (degree 3)
- $5x^6-2x^2$ (degree 6)
- 10 (degree 0)

The following are not polynomials
- $\frac{1}{x}+x^3$ (the $\frac{1}{x}$ term can be written as x^{-1} which is a **negative** power)
- $x^2+3x+\sqrt{x}$ (the $\sqrt{x}$ term can be written as $x^{\frac{1}{2}}$ which has a **non-integer** power)
- $x^5+2-\sin x$ ($\sin x$ is not a power of x)

Simplifying Fractions - one single fraction (GCSE recap)

Example 1: Splitting Fractions	Example 2: Cancelling	Example 3: Factorising and cancelling
$\frac{8x^4-4x^3+6x}{2x}$	$\frac{(x+4)(3x-1)}{3x-1}$	$\frac{x^2+3x+2}{x^2+5x+4}$
Split up terms since only 1 term in the denominator (we couldn't do this if more than 1 term in the denominator) $=\frac{8x^4}{2x}-\frac{4x^3}{2x}+\frac{6x}{2x}$ Cancel $=4x^3-2x^2+3$	Cancel common factors $3x-1$ $=\frac{(x+4)\cancel{(3x-1)}}{\cancel{3x-1}}$ $=x+4$	Factorise $\frac{(x+1)(x+2)}{(x+1)(x+4)}$ Cancel common factors $x+1$ $=\frac{\cancel{(x+1)}(x+2)}{\cancel{(x+1)}(x+4)}$ $=\frac{(x+2)}{(x+4)}$

4 Polynomial Division Methods (pick whichever works for you – method 2 is taught in schools)

Method 1: Box/Grid Method

Divide $2x^3+x^2-13x+6$ by $x+3$

Template
- Pink box needs to give $2x^3$
- Orange boxes combined need to give x^2
- Green boxes combined need to give $-13x$
- blue box needs to give 6

Once we figure what goes at the top, that is our answer

	?	?	?
x	?	?	?
$+3$	?	?	?

Step 1: Put the first term in the top left box

Step 2: Decide what needs to go in the first red ? position. We want $2x^3$ and have x hence $2x^2$

Step 3: Do the multiplication for the bottom left cell

Step 4: The orange boxes need to add to $1x^2$ and we already have $6x^2$ hence $-5x^2$ needs to go in the top middle row

Step 5: Decide what needs to go in the second red ? position. We want $-5x^2$ and have x hence $-5x$

Step 6: Do the multiplication for the bottom middle cell

Step 7: The green boxes need to add to $-13x$ and we already have $-15x$ hence $2x$ needs to go in the top right row

Step 8: Decide what needs to go in the last red ? position. We want $2x$ and have x hence 2

Step 9: Do the multiplication for the final cell

Check the constant term in the box matches the constant from the equation

$2x^3+x^2-13x+6$

	$2x^2$	$-5x$	$+2$
x	$2x^3$	$-5x^2$	$2x$
$+3$	$6x^2$	$-15x$	6

We get 6 in the box and the equation has a 6 hence matches.

Answer comes from the top of the box: $2x^2-5x+2$

What if the term in the final cell didn't match the equation?
It means we either made a mistake or there is a remainder (doesn't divide exactly). In general, the remainder is found by comparing the terms in the last column of the box with the original dividend $2x^2+x^2-13x+6$ and determining the difference. If the terms in the box don't perfectly match the original dividend, the difference represents the remainder.

Hindsight:
- Notice how like terms are on the diagonal
- There is a tooth pattern to this (down, diagonally up, down, diagonally up etc)

Method 2: Long Division

Divide $2x^3+x^2-13x+6$ by $x+3$

Set up the division using the template: divisor ⟌ dividend

$x+3\,\overline{)\,2x^3+x^2-13x+6}$

Step 1: Compare the first term of the dividend and the divisor

Step 2: Check what you need to multiply by and write at the top

Step 3: Multiply out and write under (be sure to **line up** common terms)

Step 4: Subtract and then bring down the next term afte

Repeat the 4 steps again

Step 5: Compare the first term in each

Step 6: Check what you need to multiply by and write at the top

Step 7: Multiply out and write under be sure to line up common terms)

Step 8: Subtract and then bring down the next term after

Repeat the 4 steps again

Step 9: Compare the first term in each

Step 10: Check what you need to multiply by and write at the top

Step 11: Multiply out and write under be sure to line up common terms)

Step 12: Subtract

Your final term is the remainder. If this is 0 then we say the polynomial divides exactly

Remainder is zero and we are done

```
        2x² − 5x + 2
x + 3 | 2x³ + x² − 13x + 6
        2x³ + 6x²
             −5x² − 13x
             −5x² − 15x
                     2x + 6
                     2x + 6
                          0
```

Answer comes from above the top dividing line: $2x^2-5x+2$

How did we know to stop?
We know we're done when we either get a constant upon subtracting or until we compare and can't multiply by anything since the power we have is smaller.

Summary Of Method:
Step 1: Compare the first term in each
Step 2: Check what you need to multiply by and write at the top
Step 3: Multiply out and write under (be sure to line up common terms)
Step 4: Subtract and bring down the next term after
Repeat these 4 steps until you either get a constant upon subtracting or until you compare and can't multiply by anything since the power you have is smaller. Your final term is the remainder. If this is 0 then we say the polynomial divides exactly

Method 3: Synthetic Division (aka drop the baby and carry it)

Divide $2x^3+x^2-13x+6$ by $x+3$

Step 1: Put what you're dividing by in a box and the coefficients in a line. Then leave a gap and draw a line

−3 | 2 1 −13 6

Step 2: Drop the first number right down

Step 3: Multiply. Move/carry to next column. Add. Then move to next column under 1. Then add vertically.

Step 4: Consider the next column and do the same. Multiply. Move/carry to next column. Add. Then move to next column under −13. Then add vertically.

Step 5: Consider the next column and do the same: Multiply. Move/carry to next column. Add. Then move to next column under 6. Then add vertically.

```
−3 | 2   1   −13   6
   |    −6    15  −6
   ----------------
     2  −5     2   0
```

This last number is the remainder. If it divides exactly, the remainder should be 0

These numbers are the coefficients of our answer. The right most one is the constant, then x term then the x^2 term

Answer: $2x^2-5x+2$

Method 4: Comparing coefficients

Divide $2x^3+x^2-13x+6$ by $x+3$

$cubic \equiv (linear)(something) + remainder$

Since $2x^3+x^2-13x+6$ is a cubic equation, it can be written as the multiplication of a linear and a quadratic polynomial.

$2x^3+x^2-13x+6 \equiv (x+3)(quadratic) + remainder$

$2x^3+x^2-13x+6 \equiv (x+3)(ax^2+bx+c)+R$
$RHS=(x+3)(ax^2+bx+c)+R$
$=ax^3+bx^2+cx+3ax^2+3bx+3c+R$
$=ax^3+(b+3a)x^2+(c+3b)x+3c+R$

Shortcut with 'a' and 'R': If we know it divided exactly then set $R=0$ from beginning and we can set a also early on by using the first term to guide us

Therefore:
$2x^3+x^2-13x+6 \equiv ax^3+(b+3a)x^2+(c+3b)x+3c+R$

Comparing the coefficients based on their colours:

$2x^3=ax^3$ $\Rightarrow 2=a$	$x^2=(b+3a)x^2$ $\Rightarrow 1=b+3a$ Update with a found $1=b+3(2)$ $\therefore b=-5$	$-13x=(c+3b)x$ $\Rightarrow -13=c+3b$ Update with b found $-13=c+3(-5)$ $\therefore c=2$	$6=3c+R$ Update with c found $6=3(2)+R$ $R=0$

$ax^2+bx+c+R$ becomes $2x^2-5x+2+0$

Answer: $2x^2-5x+2$

Comparisons of the 4 division methods and limitations

Box/Grid Method	Long Division	Synthetic Division	Comparing Coefficients
Very visual and a good method for weaker students	This is just like long division for numbers, but students often forget the steps (although there are only 4 on repeat) This is the most commonly used method!	**Limitation** – only works for linear factors, so if dividing by a quadratic factor we cannot use synthetic division (but for year 1 there are only linear factors) There is a **caveat** with synthetic division when dividing by form $ax+b$. For example $\frac{3x^3+12x^2-x-9}{3x-1}$ We need to turn the denominator into the form $x\pm a$. Think of this as $x^3+4x^2-\frac{1}{3}x-3$ divided by $x-\frac{1}{3}$ then proceed as normal	advantage to use comparing coefficients 0 - when we have unknown coefficients

Examples of dividing polynomials

These will all be done method 2 – long division
"Compare first with first, multiply out, subtract, bring down next term repeat"

Type 1:

Example 1	Example 2	Example 3: By a quadratic	Example 4: By a quadratic
Find the remainder when $2x^3-5x^2-16x+8$ is divided by $(x-4)$	Find the remainder when $10x^3+43x^2-2x-10$ is divided by $(5x+4)$	Divide $2x^3-3x^2-11x+6$ by $2x^2+3x-2$	Divide $3x^4-2x^3-5x^2+2x-4$ by x^2-4
Quotient $2x^2+3x-4$; $x-4 \mid 2x^3-5x^2-16x+8$; $2x^3-8x^2$; $3x^2-16x$; $3x^2-12x$; $-4x+8$; $-4x+16$; -8. remainder is -8	Quotient $2x^2+7x-6$; $5x+4 \mid 10x^3+43x^2-2x-10$; $10x^3+8x^2$; $35x^2-2x$; $35x^2+28x$; $-30x-10$; $-30x-24$; 14. remainder is 14	Quotient $x-3$; $2x^2+3x-2 \mid 2x^3-3x^2-11x+6$; $2x^3+3x^2-2x$; $-6x^2-9x+6$; $-6x^2-9x+6$; 0. Quotient is $x-3$	Quotient $3x^2-2x+7$; $x^2-4 \mid 3x^4-2x^3-5x^2+2x-4$; $3x^4-12x^2$; $-2x^3+7x^2+2x$; $-2x^3+8x$; $7x^2-6x-4$; $7x^2-28$; $-6x+24$. Quotient is $3x^2-2x+7$ Remainder is $-6x+24$

Type 2 : Missing terms

Fill in missing terms with place holders. For example: divide $4x^3-13x-6$ by $x-2$ $\Rightarrow$ write as $4x^3+0x^2-13x-6$

Example 5	Example 6	Example 7	Example 8
Divide $2x^3+9x^2+25$ by $(x+5)$	Divide x^3+x-4 by $(x-2)$	Divide x^3-1 by $(x-1)$	Divide $3x^4-2x^3-5x^2-4$ by x^2-4
Quotient $2x^2-x+5$; $x+5 \mid 2x^3+9x^2+0x+25$; $2x^3+10x^2$; $-x^2+0x$; $-x^2-5x$; $5x+25$; $5x+25$; 0. Quotient is $2x^2-x+5$ remainder is 0	Quotient x^2+2x+5; $x-2 \mid x^3+0x^2+x-4$; x^3-2x^2; $2x^2+x$; $2x^2-4x$; $5x-4$; $5x-10$; 6. Quotient is x^2+2x+5 remainder is 6	Quotient x^2+x+1; $x-1 \mid x^3+0x^2+0x-1$; x^3-x^2; x^2+0x; x^2-x; $x-1$; $x-1$; 0. Quotient is x^2+x+1 remainder is 0	Quotient $3x^2-2x+7$; $x^2-4 \mid 3x^4-2x^3-5x^2+0x-4$; $3x^4-12x^2$; $-2x^3+7x^2+0x$; $-2x^3+8x$; $7x^2-8x-4$; $7x^2-28$; $-8x+24$. Quotient is $3x^2-2x+7$ remainder is $-8x+24$

Type 3: Factorising

divisor ⟌ dividend with quotient on top becomes **dividend = (divisor)(quotient)**

Example 9	Example 10	Example 11
Find the remainder when $x^3+6x^2+5x-12$ is divided by i. $x-2$ ii. $x+3$ iii. Hence find all solutions to $x^3+6x^2+5x-12=0$	Given that $2x-1$ is a factor of $2x^3+3x^2-8x+3$ find the other 2 factors and hence factorise	$f(x)=4x^4-17x^2+4$. Divide $f(x)$ by $(2x+1)$. Give your answer in the form $f(x)=(2x+1)(ax^3+bx^2+cx+d)$
i. Quotient $x^2+8x+21$; $x-2 \mid x^3+6x^2+5x-12$; x^3-2x^2; $8x^2+5x$; $8x^2-16x$; $21x-12$; $21x-42$; 30. ii. Quotient x^2+3x-4; $x+3 \mid x^3+6x^2+5x-12$; x^3+3x^2; $3x^2+5x$; $3x^2+9x$; $-4x-12$; $-4x-12$; 0. Since $x+3$ gives remainder 0 $x^3+6x^2+5x-12=(x+3)(x^2+3x-4)$ $x^3+6x^2+5x-12=(x+3)(x+4)(x-1)$ $x=-3,-4,1$	Quotient x^2+2x-3; $2x-1 \mid 2x^3+3x^2-8x+3$; $2x^3-x^2$; $4x^2-8x$; $4x^2-2x$; $-6x+3$; $-6x+3$; 0. $(2x-1)(x^2+2x-3)$ $(2x-1)(x+3)(x-1)$ The other 2 factors are $(x+3)$ and $(x-1)$	Quotient $2x^3-x^2-8x+4$; $2x+1 \mid 4x^4+0x^3-17x^2+0x+4$; $4x^4+2x^3$; $-2x^3-17x^2$; $-2x^3-x^2$; $-16x^2+0x$; $-16x-8x$; $8x+4$; $8x+4$; 0. Quotient is $2x^3-x^2-8x+4$ $4x^4-17x^2+4=(2x+1)(2x^3-x^2-8x+4)$

Note: Factor theorem is much better to deal with these 2 types of questions above when we know a factor

Type 4: With Unknowns

Example 12:	Example 13: With unknowns	Example 14: Very hard
Given that $x+2$ and $x^2+6x+13$ are roots of x^3+ax^2+bx+c find a,b, and c	Divide x^2 by $kx-1$	The polynomial x^4+2x^3+ax+b, where a and b are constants is divisible by x^2-x+1. Find the values of a and b and the other 2 real factors
Note: This is an example that shows how it is an advantage to use comparing coefficents when we have unknown corefficients. We will multiply the factors, expand it, and then compare coefficients (this is better than dividing when there are unknown coefficients $(x+2)(x^2+6x+13)=x^3+ax^2+bx+c$ $x^3+6x^2+13x+2x^2+12x+26=x^3+ax^2+bx+c$ $x^3+8x^2+25x+26=x^3+ax^2+bx+c$ Hence, we can compare coefficients : $a=8,b=25,c=26$	Quotient $\frac{1}{k}x+\frac{1}{k^2}$; $kx-1 \mid x^2+0x+0$; x^2-x/k; $x/k+0$; $x/k-1/k^2$; $\frac{1}{k^2}$. Quotient is $\frac{1}{k}x+\frac{1}{k^2}$ remainder is $\frac{1}{k^2}$	Quotient x^2+3x+2; $x^2-x+1 \mid x^4+2x^3+0x^2+ax+b$; $x^4-x^3+x^2$; $3x^3-x^2+ax$; $3x^3-3x^2+3x$; $2x^2+(a-3)x+b$; $2x^2-2x+2$; $(a-1)x+(b-2)$. Divisible hence remainder =0 $(a-1)x+(b-2)=0$ $\Rightarrow a=1, \Rightarrow b=2$ The other two real factors: $x^2+3x+2=(x+1)(x+2)$ Note: **Comparing coefficients** would have been better than long division!

Getting Your Answer In Certain Forms : $\frac{a}{b}$ can produce 2 forms

Form 1	Form 2
$\frac{a}{b}=Q+\frac{R}{b}$	$a=Qb+R$

a = dividend
b = divisor
Q = quotient
R = remainder

We usually use form 1. Form 2 is just form 1 re-arranged to get rid of the fraction (multiplying all terms by b)

Example 15	Example 16	Example 17	Example 18
$\frac{x^3+8x^2-9x+12}{x+6}=Ax^2+Bx+C+\frac{D}{x+6}$. Find the values of A,B,C and D	$\frac{30x^3-26x^2-48x+8}{3x^2-2x-7}=Ax+B+\frac{Cx+D}{3x^2-2x-7}$. Find the values of A,B,C and D	$x^4-4x-8=(x-2)(x^3+ax^2+bx+c)+d$. Find the values of a,b,c, and d	$5x^4-4x^3+17x^2-5x+7=(Ax^2+Bx+C)(x^2+2)+Dx+E$ Find the values of A, B, C and D
Quotient $x^2+2x-21$; $x+6 \mid x^3+8x^2-9x+12$; x^3+6x^2; $2x^2-9x$; $2x^2+12x$; $-21x+12$; $5x-126$; 138. Quotient is $x^2+2x-21$ remainder is 138. $\frac{x^3+8x^2-9x+12}{x+6}=\text{Quotient}+\frac{\text{Remainder}}{\text{Divisor}}$ $=x^2+2x-21+\frac{138}{x+6}$ $\Rightarrow A=1,B=2,$ $C=-21,D=138$	Quotient $10x-2$; $3x^2-2x-7 \mid 30x^3-26x^2-48x+8$; $30x^3-20x^2-70x$; $-6x^2+22x+8$; $-6x^2+4x+14$; $18x-6$. Our quotient is $10x-2$ and remainder is $18x-6$. $\frac{30x^3-26x^2-48x+8}{3x^2-2x-7}=\text{Quotient}+\frac{\text{Remainder}}{\text{Divisor}}=$ $10x-2+\frac{18x-6}{3x^2-2x-7}\Rightarrow A=10,B=-2,C=18,D=-6$	This is another way to say do $\frac{x^4-4x-8}{x-2}$. Quotient x^3+2x^2+4x+4; $x-2 \mid x^4+0x^3+0x^2-4x-8$; x^3-2x^3; $2x^3+0x^2$; $2x^3-4x^2$; $4x^2-4x$; $4x^2-8x$; $4x-8$; $4x-8$; 0. Quotient is x^3+2x^2+4x+4 remainder is 0 $x^4-4x-8=(x^3+2x^2+4x+4)(x-2)+0$ $\Rightarrow a=2,b=4,c=4,d=0$ Note: **Comparing coefficients** would have been better than long division!	This is another way to say $\frac{5x^4-4x^3+17x^2-5x+7}{x^2+2}$. Quotient $5x^2-4x+7$; $x^2+2 \mid 5x^4-4x^3+17x^2-5x+7$; $5x^4+10x^2$; $-4x^3+7x^2-5x$; $-4x^3-8x$; $7x^2+3x+7$; $7x^2+14$; $3x-7$. Quotient is $5x^2-4x+7$ remainder is $3x-7$ $=5x^2-4x+7+\frac{3x-7}{x^2-1}$ $\Rightarrow A=5,B=-4,C=7$ $D=3,E=-7$

Factor Theorem

Why use the factor theorem? The factor theorem is a **quick way** or finding a **linear factor** of a polynomial

Definition: The factor theorem states that if $f(x)$ is a polynomial, then
- If $f(p)=0$, then $(x-p)$ is a factor
- If $(x\pm p)$ is a factor of $f(x)$, then $f(\mp p)=0$

Steps:
Step 1: set what you're dividing by equal to 0 and solve
Step 2: plug this value found into the function. The answer is the remainder. If remainder is zero, we have a special name called 'a factor.'
Step 3: We only do this step if finding an answer. Set the function from step 2 equal to the remainder (0 if given a factor or the remainder if given a remainder 0

Note: Factor theorem is a special case of remainder theorem (factor theorem is just a remainder of 0. Generally, $f(x)$ divided by $bx\pm a$ gives remainder $f\left(\mp\frac{a}{b}\right)$

Example 1: Proving a factor and Factorising/Solving	Example 2: given factor – 1 unknown
$f(x)=x^3+4x^2+x-6$ i. Use the factor theorem to show that $(x+2)$ is a factor of $f(x)$. ii. Hence factorise $f(x)$ completely. iii. Write down all the solutions to the equation $x^3+4x^2+x-6=0$	Consider $f(x)=x^3-2x^2-5x+k$. Find the value of k if $(x-1)$ is a factor of $f(x)$
i. $f(x)=x^3+4x^2+x-6$ $x+2=0$ $x=-2$ $f(-2)=0$ $f(-2)=(-2)^3+4(-2)^2+(-2)-6=0$ Hence proven, it is a factor. ii. Quotient x^2+2x-3; $x+2 \mid x^3+4x^2+x-6$; x^3+2x^2; $2x^2+x$; $2x^2+4x$; $-3x-6$; $-3x-6$; 0. iii. We now find the solutions $x^3+4x^2+x-6=0$ $(x+2)(x+3)(x-1)=0$ $x=-3,x=-2,x=1$	$f(x)=x^3-2x^2-5x+k$ $x-1=0$ $x=1$ $f(1)=0$ $f(1)=(1)^3-2(1)^2-5(1)+k=-7+k$ $-7+k=0$ $k=7$

Example 3: Given remainder – 1 unknown	Example 4: Given factors – 2 unknowns	Example 5: With hidden discriminant	Example 6: With hidden discriminant
$f(x)=ax^3+3x^2-8x+2$, where a is constant. Given that when $f(x)$ is divided by $(x-2)$ the remainder is 3, find the value of a	Given that $(x+2)$and $(2x-1)$ are factors of $f(x)=2x^3+ax^2+bx+6$ i. Find the value of a and the value of b ii.Hence find the third factor	$f(x)=3x^3-12x^2+6x-24$ i. Use the factor theorem to show that $(x-4)$ is a factor of $f(x)$ ii. Hence, show that 4 is the only real root of the equation $f(x)=0$	Given that • $f(x)=2x^3+3x^2-16x+16$ • $f(-4)=0$ i.Write $f(x)$ in the form $(x+a)Q(x)$, where a is a constant and $Q(x)$ is a quadratic expression ii.Hence prove that -4 is the only real root of the equation $f(x)=0$
$f(x)=ax^3+3x^2-8x+2$ $f(2)=3$ $f(2)=a(2)^3+3(2)^2-8(2)+2=8a-2$ $8a-2=3$ $a=\frac{5}{8}$	i. $f(-2)=2(-2)^3+a(-2)^2+b(-2)=4a-2b-1$ $\therefore 4a-2b-10=0$ $2a-b=5$ ① $f\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)^3+a\left(\frac{1}{2}\right)^2+b\left(\frac{1}{2}\right)=\frac{a}{4}+\frac{b}{2}+\frac{25}{4}$ $\frac{a}{4}+\frac{b}{2}+\frac{25}{4}=0$ $a+2b=-25$ ② Solve simultaneously: $\boldsymbol{a=-3,b=-11}$ ii. $(2x-1)(x+2)=2x^2+3x-2$ Quotient $x-3$; $2x^2+3x-2 \mid 2x^3-3x^2-11x+6$; $2x^3+3x^2-2x$; $-6x^2-9x+6$; $-6x^2-9x+6$; 0. Hence, the third factor is $(x-3)$	i. $f(x)=3x^3-12x^2+6x-24$ $f(4)=3(4)^3-12(4)^2+6(4)-24$ $f(4)=0$ Therefore a factor ii. Quotient $3x^2+6$; $x-4 \mid 3x^3-12x^2+6x-24$; $3x^3-12x^2$; $6x-24$; $6x-24$; 0. $3x^3-12x^2+6x-24=(x-4)(3x^2+6)$ $3x^2+6=0$ $b^2-4ac=(0)^2-4(3)(6)=-72<0$ Hence $3x^2+6$ no real roots $x=-4$ only root	i. Quotient $2x^2-5x+4$; $x+4 \mid 2x^3+3x^2-16x+16$; $2x^3+8x^2$; $-5x^2-16x$; $-5x^2-20x$; $4x+16$; $4x+16$; 0. $2x^3+3x^2-16x+16=(x+4)(2x^2-5x+4)$ ii. $(x+4)(2x^2-5x+4)=0$ $x+4=0$ $x=-4$ $2x^2-5x+4=0$ $b^2-4ac=(-5)^2-4(2)(4)$ $25-32=-7<0$ Hence $2x^2-5x+4$no real roots $x=-4$ only root

Example 7 given two factors – 2 unknowns + FACTORISING	Example 8: with unknowns	Example 9: with unknowns
It is given that $g(x)=3x^4+bx^3+cx^2-7x-4$ has factors $(x+1)$ and $(x-1)$. i. Find the value of b and the value of c. ii. Factorise $g(x)$ completely.	When $f(x)=2x^3+ax^2+bx-6$ is divided by $(2x-1)$ the remainder is -5 and when divided by $(x+2)$ there is no remainder. Find the values of a and b.	When the polynomial $3x^3+ax+b$ is divided by $(x-2)$ the remainder is 2, and when divided by $(x+1)$, it is 5. Find the value of a and the value of b.
i. $g(1)=3(1)^4+b(1)^3+c(1)^2-7(1)-4=b+c-8$ $b+c-8=0$ $b+c=8$ ① $g(-1)=3(-1)^4+b(-1)^3+c(-1)^2-7(-1)-4=-b+c+6$ $-b+c+6=0$ $b-c=6$ ② Solve simultaneously: $\boldsymbol{b=7,c=1}$ ii.It is quickest to divide by both factors straight away rather than each. Quotient $3x^2+7x+4$; $x^2-1 \mid 3x^4+7x^3+x^2-7x-4$; x^4-3x^2; $7x^3+4x^2-7x$; $7x^3-7x$; $4x^2+0x-4$; $4x^2-4$; 0. $3x^4+7x^3+x^2-7x-4=(x+1)(x-1)(3x^2+7x+4)$ $=(x+1)(x-1)(3x+4)(x+1)$	$f(x)=2x^3+ax^2+bx-6$ $f\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)^3+a\left(\frac{1}{2}\right)^2+b\left(\frac{1}{2}\right)-6=\frac{a}{4}+\frac{b}{2}-\frac{23}{4}$ $\frac{a}{4}+\frac{b}{2}-\frac{23}{4}=-5$ $\frac{a}{4}+\frac{b}{2}=\frac{3}{4}$ $a+2b=3$ ① $f(-2)=4a-2b-22$ $4a-2b-22=0$ $4a-2b=22$ $2a-b=11$ ② Solve simultaneously $a+2b=3$ ① $2a-b=11$ ② $\boldsymbol{a=5,b=-1}$	$f(x)=3x^3+ax+b$ $f(2)=2a+b+24$ $2a+b+24=2$ $2a+b=-22$ ① $f(-1)=3(-1)^3+a(-1)+b=-a+b-3$ $f(-1)=-a+b-3$ $-a+b-3=5$ $-a+b=8$ ② Solve simultaneously $2a+b=-22$ ① $-a+b=8$ ② $\boldsymbol{a=-10,b=-2}$

Example 10	Example 11	Example 12: Finding your own factor (factor not given)
The polynomial $f(x)=x^3+3x^2+ax+b$ leaves the same remainder when divided by $(x-2)$ as when divided by $(x+1)$. Find the value of a.	When $3x^5-ax+b$ is divided by $(x-1)$ and $(x+1)$, the remainder are equal. Given that $a,b\in\mathbb{R}$, find i.The value of a. Ii.The set of values of b	Write the polynomial $4x^3+9x^2-3x-10$ in the form $(x\pm p)(ax^2+bx+c)$
$f(x)=x^3+3x^2+ax+b$ $f(2)=(2)^3+3(2)^2+a(2)+b=2a+b+20$ $f(-1)=(-1)^3+3(-1)^2+a(-1)+b=-a+b+2$ Same remainder $f(2)=f(-1)$ $2a+b+20=-a+b+2$ $3a=-18$ $a=-6$	i. $f(1)=3(1)^5-a(1)+b=-a+b+3$ $f(-1)=3(-1)^5-a(-1)+b=a+b-3$ $f(1)=f(-1)$ $-a+b+3=a+b-3$ $2a=6$ $\boldsymbol{a=3}$ ii. By observation, any value of b makes the equation true, hence b can be any real number $\boldsymbol{b\in\mathbb{R}}$	Check factors of the final term and see which give you 0, So we need to check factors of 10 $f(\pm1), f(\pm2), f(\pm5)$ and $f(\pm10)$. Normally the first 4 are the max we need to check $f(-2)=0$ Quotient $4x^2+x-5$; $x+2 \mid 4x^3+9x^2-3x-10$; $4x^3+8x^2$; x^2-3x; x^2+2x; $-5x-10$; $-5x-10$; 0. $(x+2)(4x^2+x-5)$

Example 12 : Quadratic factors (no repeated root)	Example 13: Quadratic factors (repeated root)	Example 14 (Hard)
Show that x^2-4 is a factor of $x^4-5x^3+2x^2+20x-24$	Show that x^2-2x+1 is a factor of x^3-x^2-x+1	Let $P(x)=2x^4-15x^3+ax^2+bx+c$, where $a,b,c\in\mathbb{R}$. i. Given that $(x-5)$ is a factor of $P(x)$, find a relationship between a, b and c. ii. Given that $(x-5)^2$ is a factor of $P(x)$, and that $a=2$, find the values of b and c
Way 1: Long division and show remainder is zero Quotient x^2-5x+6; $x^2-4 \mid x^4-5x^3+2x^2+20x-24$; x^4-4x^2; $-5x^3+6x^2+20x$; $-5x^3+20x$; $6x^2+0x-24$; $6x^2-24$; 0. Remainder is 0 hence a factor. **Way 2 Factor Theorem** $x^2-4=(x+2)(x-2)$ $f(x)=x^4-5x^3+2x^2+20x-24$ $f(-2)=(-2)^4-5(-2)^3+2(-2)^2+20(-2)-24=0$ $f(2)=(2)^4-5(2)^3+2(2)^2+20(2)-24=0$ Both are factors hence x^2-4 is a factor	Since a repeated factor need to use polynomial division. Quotient $x+1$; $x^2-2x+1 \mid x^3-x^2-x+1$; x^3-2x^2+x; x^2-2x+1; x^2-2x+1; 0. Remainder is 0 hence a factor	i. $P(x)=2x^4-15x^3+ax^2+bx+c$ $P(5)=2(5)^4-15(5)^3+a(5)^2+b(5)+c=25a+5b+c-625$ So, the value of $P(5)$ is the remainder of 0 $P(5)=25a+5b+c-625$ $25a+5b+c-625=0$ $25a+5b+c=625$ ii. From part (ii), we found an equation for $P(x)$ $P(x)=(x-5)^2(px^2+qx+r)$ $P(x)=(x^2-10x+25)(px^2+qx+r)$ $P(x)=px^4+qx^3+rx^2-10px^3-10qx^2-10rx+25px^2+25qx+25r$ We collect like terms $P(x)=px^4+(q-10p)x^3+(r-10q+25p)x^2+(-10r+25q)x+25r$ Equate this to LHS $P(x)=2x^4-15x^3+2x^2+bx+c$ Hence, $p=2$ $(q-10p)=-15\Rightarrow q-10(2)=-15\Rightarrow q=5$ $(r-10q+25p)=2\Rightarrow(r-10(5)+25(2))=2\Rightarrow r=2$ $(-10r+25q)=b\Rightarrow b=-10(2)+25(5)=105$ $25r=c=25(2)=50$ $\boldsymbol{b=105,c=50}$